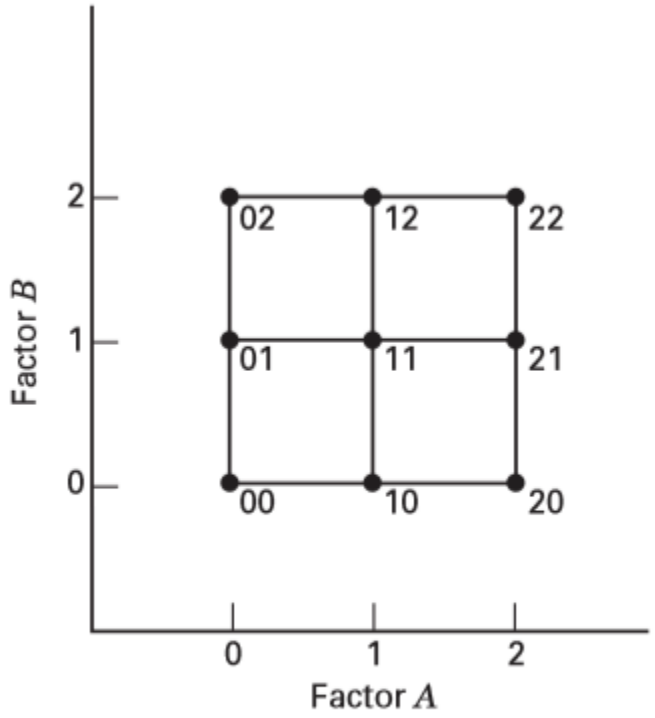
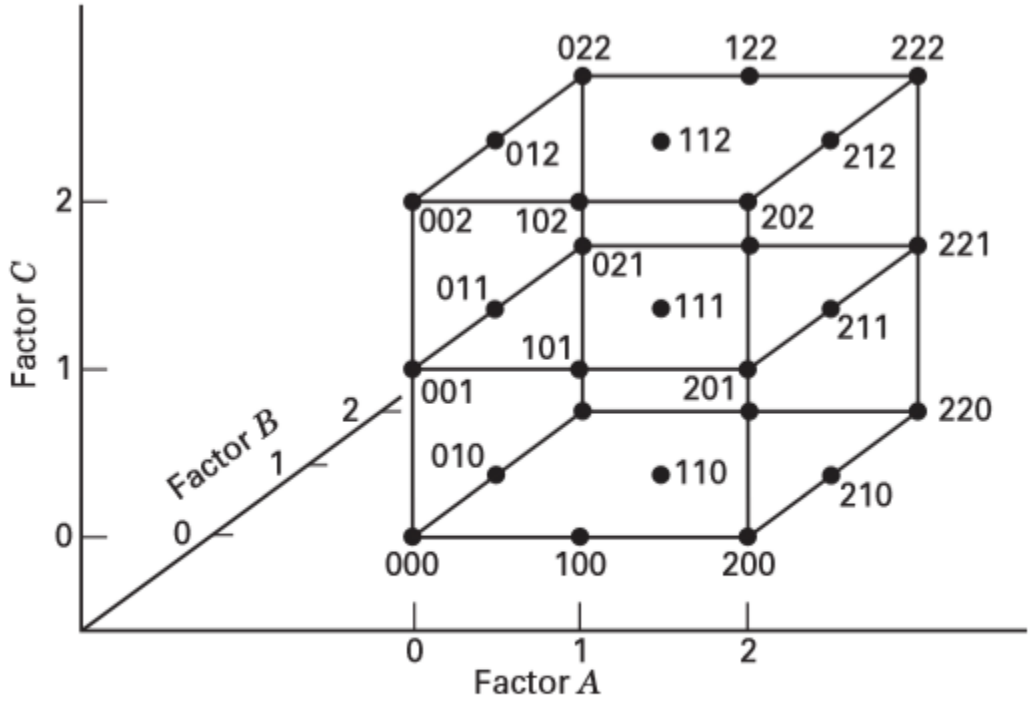
Additional Design and Analysis Topics for Factorial and Fractional Factorial Designs

- 3^k factorial design—a factorial arrangement with k factors, each at three levels
- Three levels of the factors are low, intermediate, and high.
- Each treatment combination in the 3^k design will be denoted by k digits, where the first digit indicates the level of factor A, the second digit indicates the level of factor B, . . . , and the kth digit indicates the level of factor K.
- For example, in a 3^2 design, 00 denotes the treatment combination corresponding to A and B both at the low level, and 01 denotes the treatment combination corresponding to A at the low level and B at the intermediate level



■ **FIGURE 9.1** Treatment combinations in a 3^2 design



■ **FIGURE 9.2** Treatment combinations in a 3^3 design

- In the 3^k system of designs, when the factors are quantitative, we often denote the low, intermediate, and high levels by -1, 0, and +1, respectively.
- This facilitates fitting a regression model relating the response to the factor levels.
- For example, consider the 3^2 design in Figure 9.1, and let x_1 represent factor A and x_2 represent factor B. A regression model relating the response y to x_1 and x_2 that is supported by this design is:

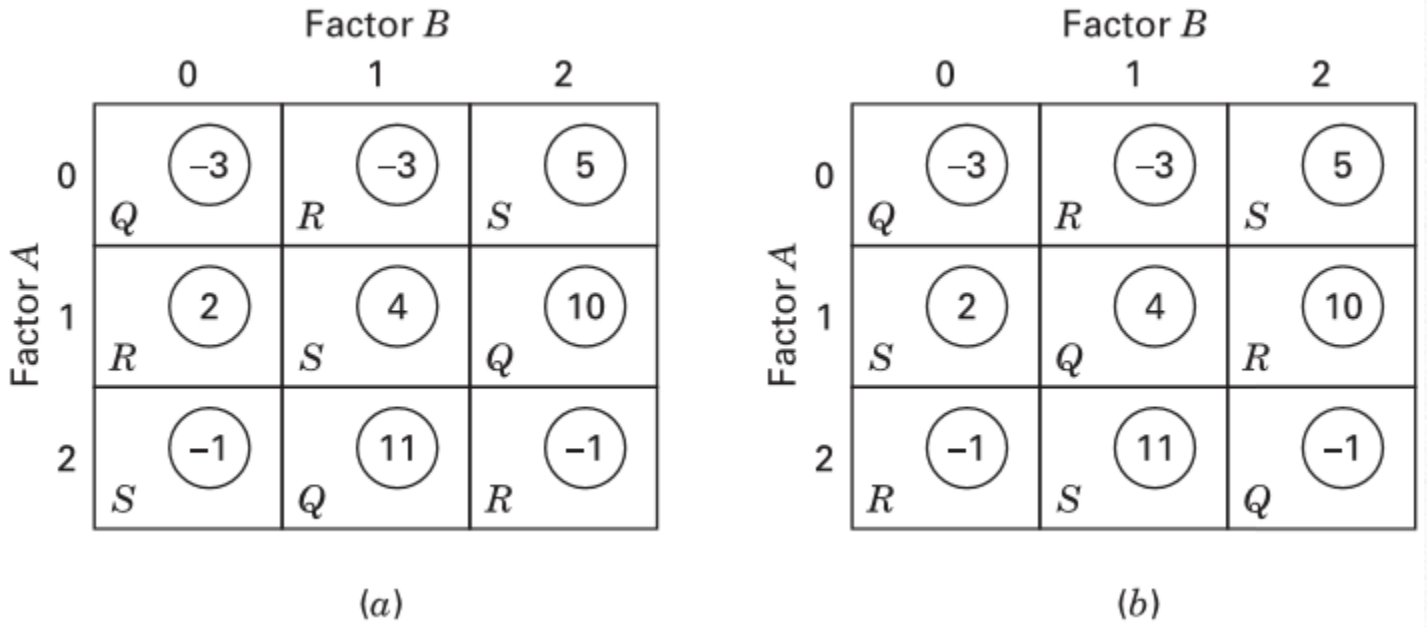
$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 + \beta_{11} x_1^2 + \beta_{22} x_2^2 + \epsilon$$

The 3^2 Design

- The simplest design in the 3^k system is the 3^2 design, which has two factors, each at three levels.
- Because there are $3^2 = 9$ treatment combinations, there are eight degrees of freedom between these treatment combinations.
- The main effects of A and B each have two degrees of freedom, and the AB interaction has four degrees of freedom. If there are n replicates, there will be $n3^2 - 1$ total degrees of freedom and $3^2(n - 1)$ degrees of freedom for error.

- The two-factor interaction AB may be partitioned in two ways. Suppose that both factors A and B are quantitative.
- The first method consists of subdividing AB into the four single-degree-of-freedom components corresponding to $AB_{L\times L}$, $AB_{L\times Q}$, $AB_{Q\times L}$, and $AB_{Q\times Q}$.
- The second method is based on orthogonal Latin squares
  - It is usually associated with the case where all factors are qualitative
- As example let us consider the totals of the treatment combinations for the data in Example 5.5.

■ **FIGURE 9.3**
Treatment combination totals from Example 5.5 with two orthogonal Latin squares superimposed



- The two factors A and B correspond to the rows and columns, respectively, of a 3 * 3 Latin square.
- In Figure 9.3, two particular 3* 3 Latin squares are shown superimposed on the cell totals.
- These two Latin squares are orthogonal; that is, if one square is superimposed on the other, each letter in the first square will appear exactly once with each letter in the second square.
- The totals for the letters in the (a) square are Q = 18, R =- 2, and S = 8, and the sum of squares between these totals is $[18^2 + (-2)^2 + 8^2]/(3)(2) - [24^2/(9)(2)] = 33.34$, two degrees of freedom.

- The letter totals in the (b) square are Q = 0, R = 6, and S = 18, and the sum of squares between these totals is $[0^2 + 6^2 + 18^2]/(3)(2) - [24^2/(9)(2)] = 28$ with two degrees of freedom
- The sum of these two components is 33.34 + 28.00 = 61.34 = SS_{AB} with 2 + 2 = 4 degrees of freedom.
- In general, the sum of squares computed from square (a) is called the AB component of interaction, and the sum of squares computed from square (b) is called the AB^2 component of interaction.
  - The components AB and AB^2 each have two degrees of freedom.

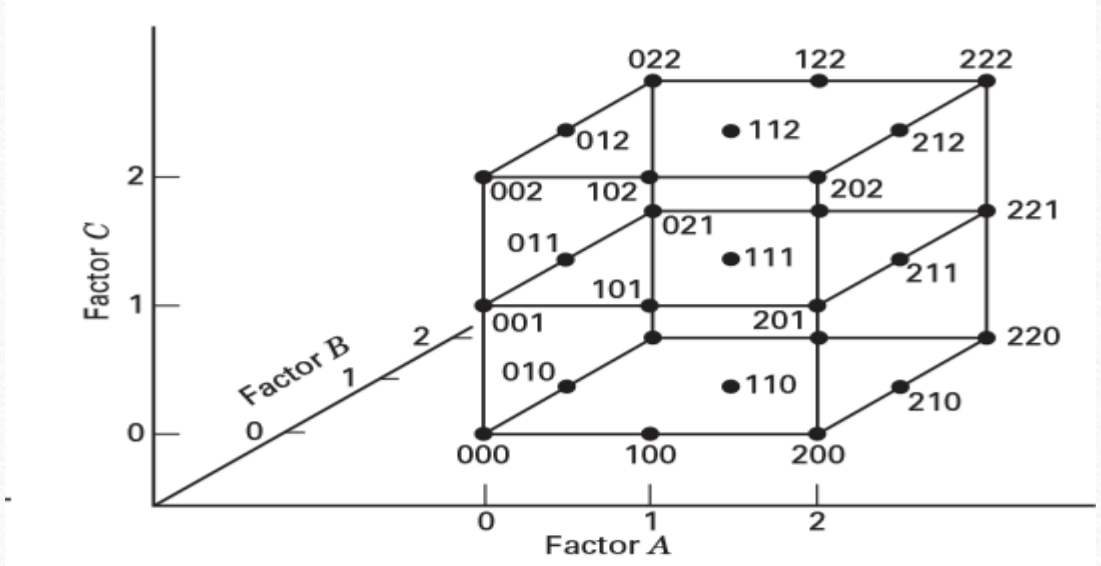
The AB and AB^2 components of interaction may be computed another way. Consider the treatment combination totals in either square in Figure 9.3. If we add the data by diagonals downward from left to right, we obtain the totals $-3 + 4 - 1 = 0$, $-3 + 10 - 1 = 6$, and $5 + 11 + 2 = 18$. The sum of squares between these totals is 28.00 (AB^2). Similarly, the diagonal totals downward from right to left are $5 + 4 - 1 = 8$, $-3 + 2 - 1 = -2$, and $-3 + 11 + 10 = 18$. The sum of squares between these totals is 33.34 (AB). Yates called these components of interaction as the ***I* and *J* components of interaction**, respectively. We use both notations interchangeably; that is,

$$I(AB) = AB^2$$

$$J(AB) = AB$$

The 3^3 Design

- Suppose there are three factors (A, B, and C) under study and that each factor is at three levels arranged in a factorial experiment.
- This is a 3^3 factorial design
-



■ FIGURE 9.2 Treatment combinations in a 3^3 design

- The 27 treatment combinations have 26 degrees of freedom.
- Each main effect has two degrees of freedom, each two factor interaction has four degrees of freedom, and the three-factor interaction has eight degrees of freedom.
- If there are n replicates, there are $n3^3 - 1$ total degrees of freedom and $3^3(n$ 1) degrees of freedom for error.

The three-factor interaction ABC may be partitioned into four orthogonal two-degrees-of-freedom components, which are usually called the W, X, Y, and Z components of the interaction. They are also referred to as the AB^2C^2, AB^2C, ABC^2, and ABC components of the ABC interaction, respectively. The two notations are used interchangeably; that is,

$$W(ABC) = AB^2C^2$$

$$X(ABC) = AB^2C$$

$$Y(ABC) = ABC^2$$

$$Z(ABC) = ABC$$

EXAMPLE 9.1

A machine is used to fill 5-gallon metal containers with soft drink syrup. The variable of interest is the amount of syrup loss due to frothing. Three factors are thought to influence frothing: the nozzle design (A), the filling speed (B), and the operating pressure (C). Three nozzles, three filling speeds, and three pressures are chosen, and two replicates of a 3^3 factorial experiment are run. The coded data are shown in Table 9.1.

Syrup Loss Data for Example 9.1 (units are cubic centimeters −70)

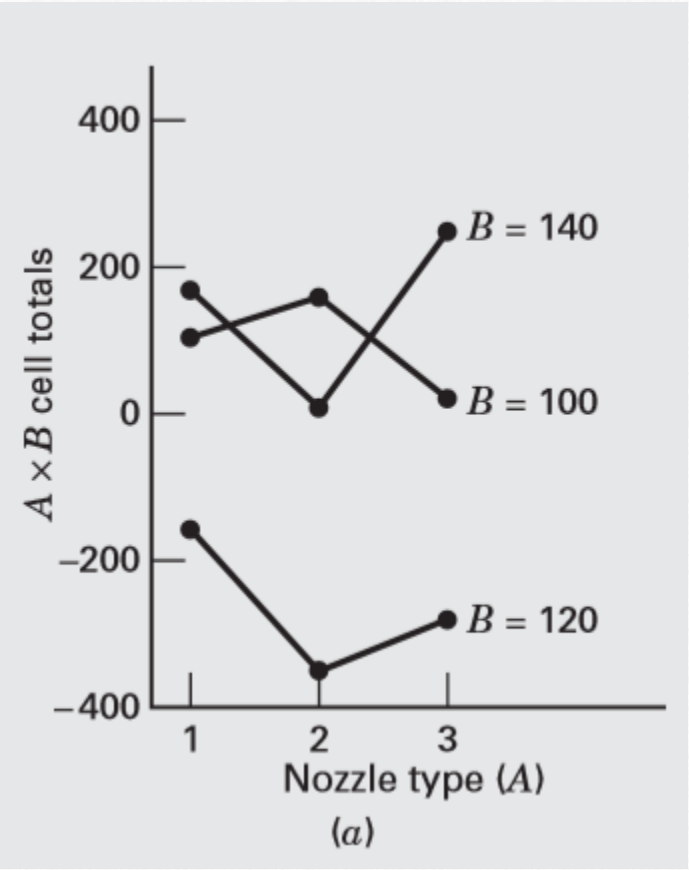
Pressure (in psi) (C)	Nozzle Type (A)								
	1			2			3		
	Speed (in RPM) (B)								
	100	120	140	100	120	140	100	120	140
10	−35	−45	−40	17	−65	20	−39	−55	15
	−25	−60	15	24	−58	4	−35	−67	−30
15	110	−10	80	55	−55	110	90	−28	110
	75	30	54	120	−44	44	113	−26	135
20	4	−40	31	−23	−64	−20	−30	−61	54
	5	−30	36	−5	−62	−31	−55	−52	4

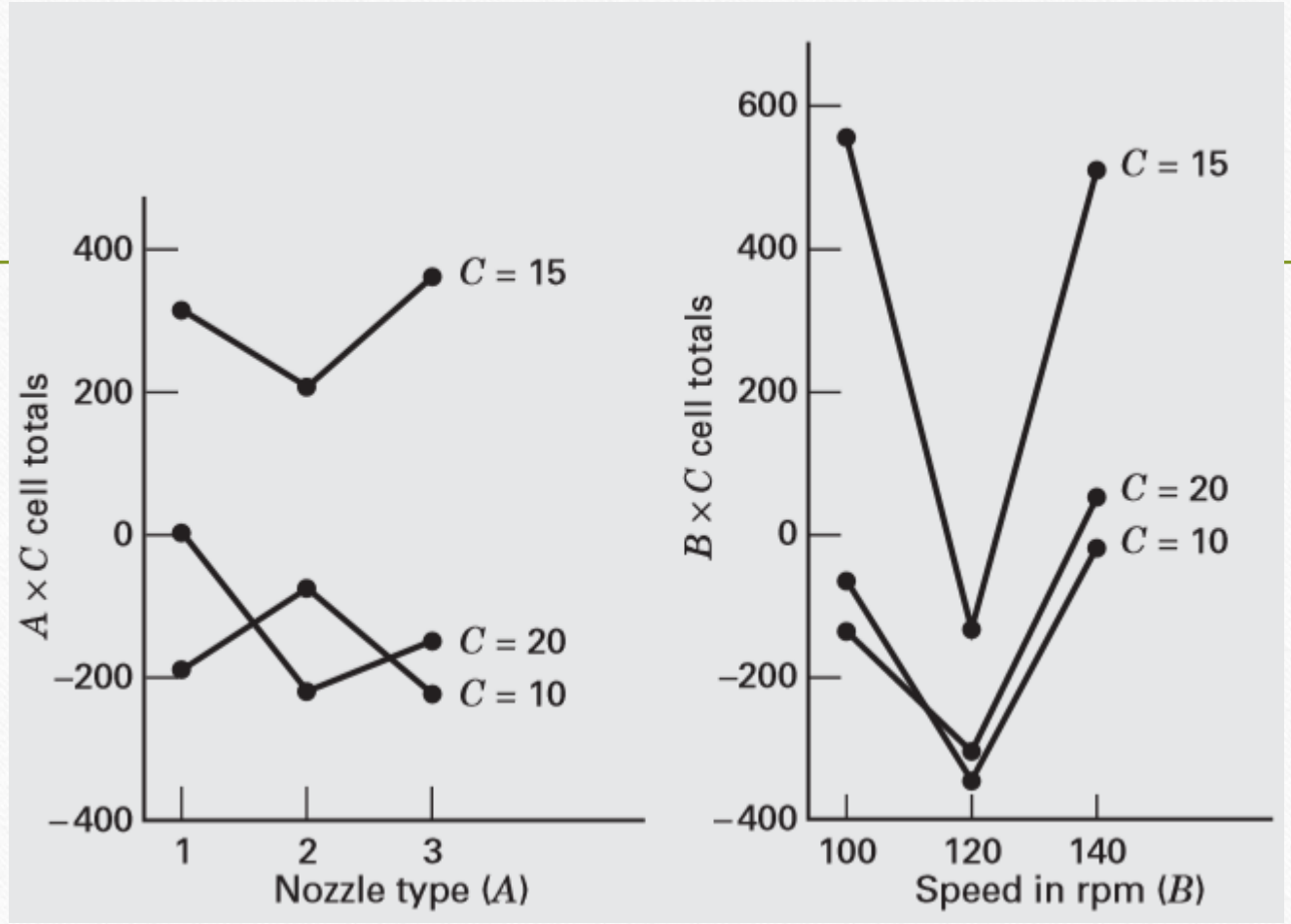
The analysis of variance for the syrup loss data is shown in Table 9.2. The sums of squares have been computed by the usual methods. We see that the filling speed and operating pressure are statistically significant. All three two-factor interactions are also significant. The two-factor interactions are analyzed graphically in Figure 9.4. The middle level of speed gives the best performance, nozzle types 2 and 3, and either the low (10 psi) or high (20 psi) pressure seems most effective in reducing syrup loss.

■ **TABLE 9.2**

Analysis of Variance for Syrup Loss Data

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	*P*-Value
A, nozzle	993.77	2	496.89	1.17	0.3256
B, speed	61,190.33	2	30,595.17	71.74	<0.0001
C, pressure	69,105.33	2	34,552.67	81.01	<0.0001
AB	6,300.90	4	1,575.22	3.69	0.0160
AC	7,513.90	4	1,878.47	4.40	0.0072
BC	12,854.34	4	3,213.58	7.53	0.0003
ABC	4,628.76	8	578.60	1.36	0.2580
Error	11,515.50	27	426.50		
Total	174,102.83	53			




400
200
0
−200
−400
$A \times C$ cell totals
$C = 15$
$C = 20$
$C = 10$
1
2
3
Nozzle type (A)
600
400
200
0
−200
−400
$B \times C$ cell totals
$C = 15$
$C = 20$
$C = 10$
100
120
140
Speed in rpm (B)

- The three-level design often finds some application; one or more of the factors are qualitative, naturally taking on three levels, and the remaining factors are quantitative.
- In this example, suppose only three nozzle designs are of interest. This is clearly, then, a qualitative factor that requires three levels.
- The filling speed and the operating pressure are quantitative factors. Therefore, we could fit a quadratic model in the two factors speed and pressure at each level of the nozzle factor.

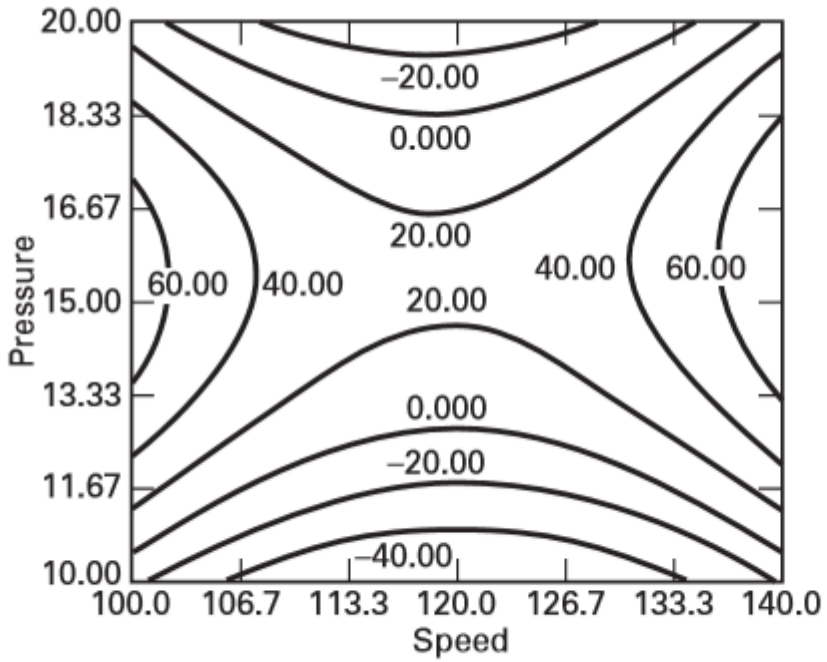
Regression Models for Example 9.1

Nozzle Type	x_1 = Speed (S), x_2 = Pressure (P) in Coded Units
1	$\hat{y} = 22.1 + 3.5x_1 + 16.3x_2 + 51.7x_1^2 - 71.8x_2^2 + 2.9x_1x_2$ $\hat{y} = 1217.3 - 31.256S + 86.017P + 0.12917S^2 - 2.8733P^2 + 0.02875SP$
2	$\hat{y} = 25.6 - 22.8x_1 - 12.3x_2 + 14.1x_1^2 - 56.9x_2^2 - 0.7x_1x_2$ $\hat{y} = 180.1 - 9.475S + 66.75P + 0.035S^2 - 2.2767P^2 - 0.0075SP$
3	$\hat{y} = 15.1 + 20.3x_1 + 5.9x_2 + 75.8x_1^2 - 94.9x_2^2 + 10.5x_1x_2$ $\hat{y} = 1940.1 - 46.058S + 102.48P + 0.18958S^2 - 3.7967P^2 + 0.105SP$

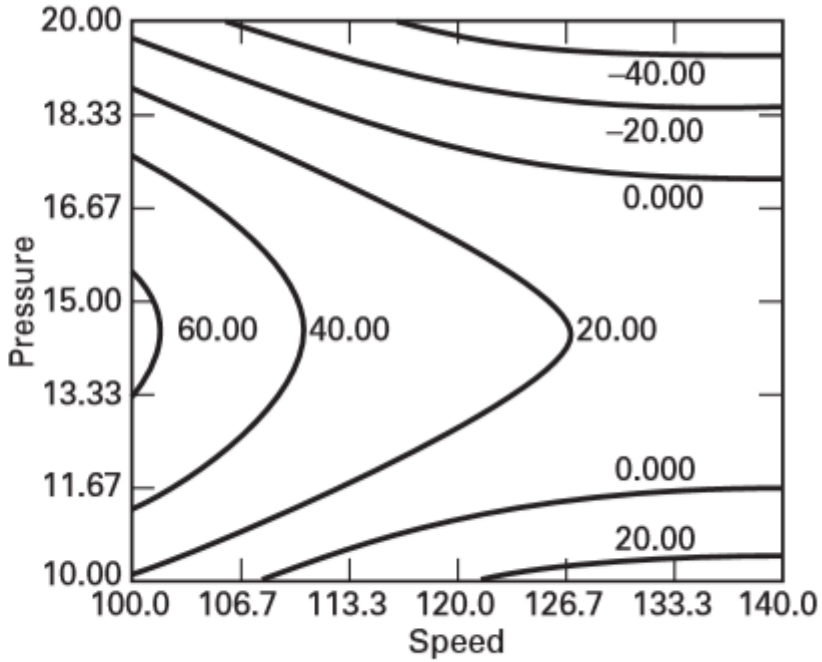
Coded Level	Speed (psi)	Pressure (rpm)
−1	100	10
0	120	15
+1	140	20

- Next we observe the response surface contour plots of constant syrup loss as a function of speed and pressure for each nozzle type.
- These plots reveal considerable useful information about the performance of this filling system. Because the objective is to minimize syrup loss, nozzle type 3 would be preferred, as the smallest observed contours (-60) appear only on this plot.
- Filling speed near the middle level of 120 rpm and the either low or high pressure levels should be used.

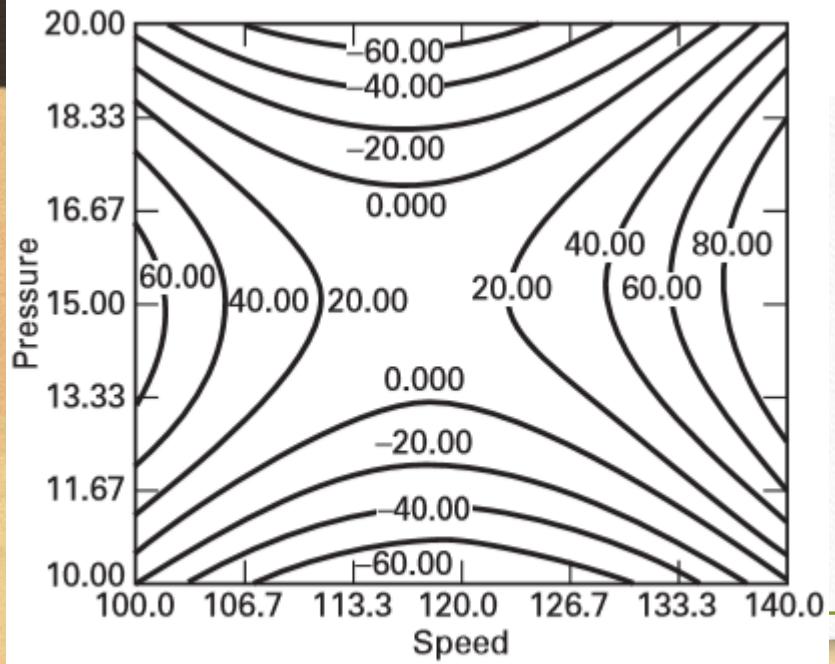
■ **FIGURE 9.5**
Contours of constant syrup loss (units: cc −70) as a function of speed and pressure for nozzle types 1, 2, and 3, Example 9.1



(*a*) Nozzle type 1



(*b*) Nozzle type 2



(*c*) Nozzle type 3

- When constructing contour plots for an experiment that has a mixture of quantitative and qualitative factors, it is not unusual to find that the shapes of the surfaces in the quantitative factors are very different at each level of the qualitative factors.
- This is noticeable to some degree in Figure 9.5, where the shape of the surface for nozzle type 2 is considerably elongated in comparison to the surfaces for nozzle types 1 and 3.
- When this occurs, it implies that there are interactions present between the quantitative and qualitative factors, and as a result, the optimum operating conditions (and other important conclusions) in terms of the quantitative factors are very different at each level of the qualitative factors.

- The numerical partitioning of the ABC interaction into its four orthogonal two-degrees-of-freedom components using the data in Example 9.1.
- First, select any two of the three factors, say AB, and compute the I and J totals of the AB interaction at each level of the third factor C.

		A			Totals	
C	B	1	2	3	I	J
	100	−60	41	−74	−198	−222
10	120	−105	−123	−122	−106	−79
	140	−25	24	−15	−155	−158
	100	185	175	203	331	238
15	120	20	−99	−54	255	440
	140	134	154	245	377	285
	100	9	−28	−85	−59	−144
20	120	−70	−126	−113	−74	−40
	140	67	−51	58	−206	−155

The $I(AB)$ and $J(AB)$ totals are now arranged in a two-way table with factor C, and the I and J diagonal totals of this new display are computed as follows:

				Totals						Totals	
C		$I(AB)$		I	J	C		$J(AB)$		I	J
10	−198	−106	−155	−149	41	10	−222	−79	−158	63	138
15	331	255	377	212	19	15	238	440	285	62	4
20	−59	−74	−206	102	105	20	−144	−40	−155	40	23

The I and J diagonal totals computed above are actually the totals representing the quantities $I[I(AB) \times C] = AB^2C^2$, $J[I(AB) \times C] = AB^2C$, $I[J(AB) \times C] = ABC^2$, and $J[J(AB) \times C] = ABC$ or the W, X, Y, and Z components of ABC. The sums of squares are found in the usual way; that is,

$$I[I(AB) \times C)] = AB^2C^2 = W(ABC)$$
$$= \frac{(-149)^2 + (212)^2 + (102)^2}{18} - \frac{(165)^2}{54} = 3804.11$$
$$J[I(AB) \times C] = AB^2C = X(ABC)$$
$$= \frac{(41)^2 + (19)^2 + (105)^2}{18} - \frac{(165)^2}{54} = 221.77$$

$$J[J(AB) \times C] = ABC^2 = Y(ABC)$$
$$= \frac{(63)^2 + (62)^2 + (40)^2}{18} - \frac{(165)^2}{54} = 18.77$$
$$J[J(AB) \times C] = ABC = Z(ABC)$$
$$= \frac{(138)^2 + (4)^2 + (23)^2}{18} - \frac{(165)^2}{54} = 584.11$$

Although this is an orthogonal partitioning of SS_{ABC}, we point out again that it is not customarily displayed in the analysis of variance table. In subsequent sections, we discuss the occasional need for the computation of one or more of these components.

The General 3^k Design

- The concepts utilized in the 3^2 and 3^3 designs can be readily extended to the case of k factors, each at three levels, that is, to a 3^k factorial design.
- The usual digital notation is employed for the treatment combinations, so 0120 represents a treatment combination in a 3^4 design with A and D at the low levels, B at the intermediate level, and C at the high level.
- There are 3k treatment combinations, with 3^k-1 degrees of freedom between them. These treatment combinations allow sums of squares to be determined for k main effects, each with two degrees of freedom; $\binom{K}{2}$ two-factor interactions, each with four degrees of freedom; . . . ; and one k-factor interaction with 2^k degrees of freedom.
- In general, an h-factor interaction has 2^h degrees of freedom. If there are n replicates, there are $n3^k$ - 1 total degrees of freedom and $3^k(n - 1)$ degrees of freedom for error.
- The size of the design increases rapidly with k. For example, a 3^3 design has 27 treatment combinations per replication, a 3^4 design has 81, a 3^5 design has 243, and so on. Therefore, only a single replicate of the 3^k design is frequently considered, and higher order interactions are combined to provide an estimate of error.

Confounding in the 3^k Factorial Design

- Even when a single replicate of the 3^k design is considered, the design requires so many runs that it is unlikely that all 3^k runs can be made under uniform conditions.
- Thus, confounding in blocks is often necessary. The 3^k design may be confounded in 3^p incomplete blocks, where $p < k$.
- Thus, these designs may be confounded in three blocks, nine blocks, and so on.

The 3^k Factorial Design in Three Block

- We wish to confound the 3^k design in three incomplete blocks.
- These three blocks have two degrees of freedom among them; thus, there must be two degrees of freedom confounded with blocks.
- 3^k factorial series each main effect has two degrees of freedom. Furthermore, every two-factor interaction has four degrees of freedom and can be decomposed into two components of interaction (e.g., AB and AB^2), each with two degrees of freedom; every three-factor interaction has eight degrees of freedom and can be decomposed into four components of interaction (e.g.,ABC,ABC^2,AB^2C,and AB^2C^2),each with two degrees of freedom; and so on.
- It is thus convenient to confound a component of interaction with blocks

The general procedure is to construct a **defining contrast**

$$L = \alpha_1 x + \alpha_2 x_2 + \cdots + \alpha_k x_k \tag{9.2}$$

where α_i represents the exponent on the ith factor in the effect to be confounded and x_i is the level of the ith factor in a particular treatment combination. For the 3^k series, we have $\alpha_i = 0$, 1, or 2 with the first nonzero α_i being unity, and $x_i = 0$ (low level), 1 (intermediate level), or 2 (high level). The treatment combinations in the 3^k design are assigned to blocks based on the value of L (mod 3). Because L (mod 3) can take on only the values 0, 1, or 2, three blocks are uniquely defined. The treatment combinations satisfying $L = 0$ (mod 3) constitute the **principal block**. This block will always contain the treatment combination 00 . . . 0.

suppose we wish to construct a 3^2 factorial design in three blocks. Either component of the AB interaction, AB or AB^2, may be confounded with blocks. Arbitrarily choosing AB^2, we obtain the defining contrast

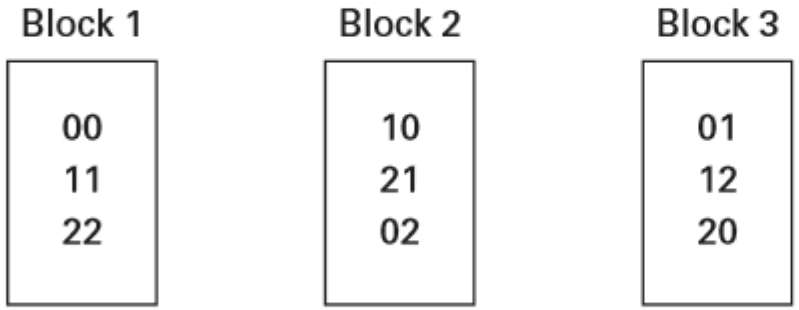
$$L = x_1 + 2x_2$$

The value of L (mod 3) of each treatment combination may be found as follows:

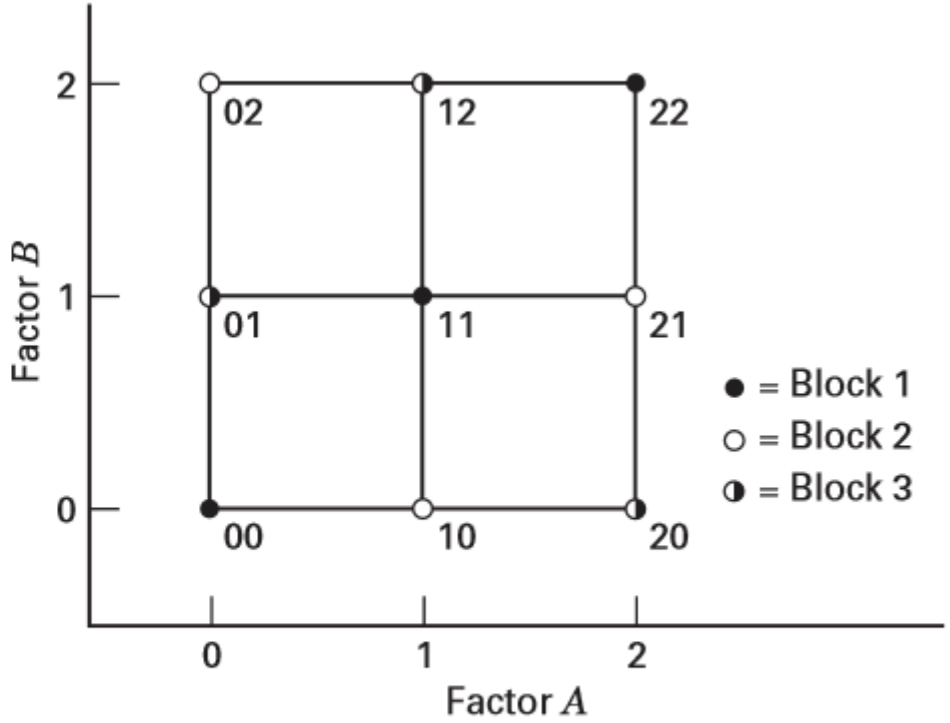
$$\begin{aligned}
00: &\quad L = 1(0) + 2(0) = 0 = 0 \pmod 3 & 11: &\quad L = 1(1) + 2(1) = 3 = 0 \pmod 3 \\
01: &\quad L = 1(0) + 2(1) = 2 = 2 \pmod 3 & 21: &\quad L = 1(2) + 2(1) = 4 = 1 \pmod 3 \\
02: &\quad L = 1(0) + 2(2) = 4 = 1 \pmod 3 & 12: &\quad L = 1(1) + 2(2) = 5 = 2 \pmod 3 \\
10: &\quad L = 1(1) + 2(0) = 1 = 1 \pmod 3 & 22: &\quad L = 1(2) + 2(2) = 6 = 0 \pmod 3
\end{aligned}$$

$$20: \quad L = 1(2) + 2(0) = 2 = 2 \pmod 3$$

■ **FIGURE 9.6**
The 3^2 design in three blocks with AB^2 confounded



(*a*) Assignment of the treatment combinations to blocks



(*b*) Geometric view

- The elements in the principal block form a group with respect to addition modulus 3.
- Referring to Figure 9.6,we see that 11 + 11 = 22 and 11 + 22 = 00. Treatment combinations in the other two blocks may be generated by adding, modulus 3, any element in the new block to the elements of the principal block. Thus, we use 10 for block 2 and obtain
  - 10 + 00 = 10 10 + 11 = 21 and 10 + 22 = 02
- To generate block 3, we find using 01
  - 01 + 00 = 01 01 + 11 = 12 and 01 + 22 = 20

EXAMPLE 9.2

We illustrate the statistical analysis of the 3^2 design confounded in three blocks by using the following data, which come from the single replicate of the 3^2 design shown in Figure 9.6.

	Block 1	Block 2	Block 3
	00 = 4	10 = −2	01 = 5
	11 = −4	21 = 1	12 = −5
	22 = 0	02 = 8	20 = 0
Block totals =	0	7	0

Using conventional methods for the analysis of factorials, we find that $SS_A = 131.56$ and $SS_B = 0.22$.

We also find that

$$SS_{\text{Blocks}} = \frac{(0)^2 + (7)^2 + (0)^2}{3} - \frac{(7)^2}{9} = 10.89$$

However, SS_{Blocks} is exactly equal to the AB^2 component of interaction. To see this, write the observations as follows:

		Factor B		
		0	**1**	**2**
Factor A	0	4	5	8
	1	−2	−4	−5
	2	0	1	0

Recall from Section 9.1.2 that the I or AB^2 component of the AB interaction may be found by computing the sum of squares between the left-to-right diagonal totals in the above layout. This yields

$$SS_{AB^2} = \frac{(0)^2 + (0)^2 + (7)^2}{3} - \frac{(7)^2}{9} = 10.89$$

which is identical to SS_{Blocks}.

The analysis of variance is shown in Table 9.4. Because there is only one replicate, no formal tests can be performed. It is not a good idea to use the AB component of interaction as an estimate of error.

■ **TABLE 9.4**

Analysis of Variance for Data in Example 9.2

Source of Variation	Sum of Squares	Degrees of Freedom
Blocks (AB^2)	10.89	2
A	131.56	2
B	0.22	2
AB	2.89	2
Total	145.56	8

We now look at a slightly more complicated design—a 3^3 factorial confounded in three blocks of nine runs each. The AB^2C^2 component of the three-factor interaction will be confounded with blocks. The defining contrast is

$$L = x_1 + 2x_2 + 2x_3$$

It is easy to verify that the treatment combinations 000, 012, and 101 belong in the principal block. The remaining runs in the principal block are generated as follows:

(1) 000	(4) 101 + 101 = 202	(7) 101 + 021 = 122
(2) 012	(5) 012 + 012 = 021	(8) 012 + 202 = 211
(3) 101	(6) 101 + 012 = 110	(9) 021 + 202 = 220

To find the runs in another block, note that the treatment combination 200 is not in the principal block. Thus, the elements of block 2 are

(1) 200 + 000 = 200	(4) 200 + 202 = 102	(7) 200 + 122 = 022
(2) 200 + 012 = 212	(5) 200 + 021 = 221	(8) 200 + 211 = 111
(3) 200 + 101 = 001	(6) 200 + 110 = 010	(9) 200 + 220 = 120

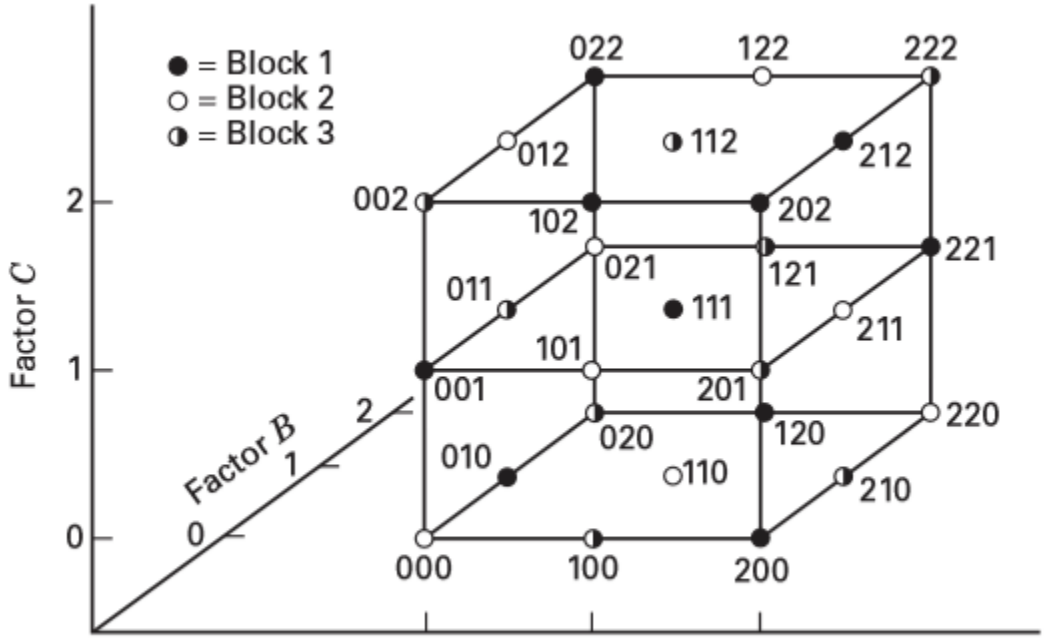
Notice that all these runs satisfy $L = 2 \pmod 3$. The final block is found by observing that 100 does not belong in block 1 or 2. Using 100 as above yields

(1) 100 + 000 = 100	(4) 100 + 202 = 002	(7) 100 + 122 = 222
(2) 100 + 012 = 112	(5) 100 + 021 = 121	(8) 100 + 211 = 011
(3) 100 + 101 = 201	(6) 100 + 110 = 210	(9) 100 + 220 = 020

■ **FIGURE 9.7**
The 3^3 design in three blocks with AB^2C^2 confounded

Block 1	Block 2	Block 3
000	200	100
012	212	112
101	001	201
202	102	002
021	221	121
110	010	210
122	022	222
211	111	011
220	120	020

(*a*) Assignment of the treatment combinations to blocks



(*b*) Geometric view

- The analysis of variance for this design is shown in Table 9.5(in next slide).
- Through the use of this confounding scheme, information on all the main effects and two-factor interactions is available. The remaining components of the three-factor interaction (ABC, AB^2C, and ABC^2) are combined as an estimate of error.
- The sum of squares for those three components could be obtained by subtraction. In general, for the 3^k design in three blocks, we would always select a component of the highest order interaction to confound with blocks.
- The remaining unconfounded components of this interaction could be obtained by computing the k-factor interaction in the usual way and subtracting from this quantity the sum of squares for blocks.

■ **TABLE 9.5**

Analysis of Variance for a 3^3 Design with AB^2C^2 Confounded

Source of Variation	Degrees of Freedom
Blocks (AB^2C^2)	2
A	2
B	2
C	2
AB	4
AC	4
BC	4
Error ($ABC + AB^2C + ABC^2$)	6
Total	26

The 3^k Factorial Design in Nine Blocks

- We choose two components of interaction, and, as a result, two more will be confounded automatically, yielding the required eight degrees of freedom.

- These two are the generalized interactions of the two effects originally chosen. In the 3^k system, the generalized interactions of two effects (e.g., P and Q) are defined as PQ and PQ^2 (or P^2Q).

The two components of interaction initially chosen yield *two* defining contrasts

$$L_1 = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k = u \pmod 3 \qquad u = 0, 1, 2$$

$$L_2 = \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_k x_k = h \pmod 3 \qquad h = 0, 1, 2 \qquad \textbf{(9.3)}$$

where $\{\alpha_i\}$ and $\{\beta_j\}$ are the exponents in the first and second generalized interactions, respectively, with the convention that the first nonzero α_i and β_j are unity. The defining contrasts in Equation 9.3 imply nine simultaneous equations specified by the pair of values for L_1 and L_2. Treatment combinations having the same pair of values for (L_1, L_2) are assigned to the same block.

- The principal block consists of treatment combinations satisfying L1 = L2 = 0 (mod 3). The elements of this block form a group with respect to addition modulus 3
- Consider the 3^4 factorial design confounded in nine blocks of nine runs each. Suppose we choose to confound ABC and AB^2D^2.
- Their generalized interactions:

$$(ABC)(AB^2D^2) = A^2B^3CD^2 = (A^2B^3CD^2)^2 = AC^2D$$

$$(ABC)(AB^2D^2)^2 = A^3B^5CD^4 = B^2CD = (B^2CD)^2 = BC^2D^2$$

  - Are also confounded with blocks

The defining contrasts for ABC and AB^2D^2 are

$$L_1 = x_1 + x_2 + x_3$$

$$L_2 = x_1 + 2x_2 + 2x_4 \qquad (9.4)$$

- For the 3k design in nine blocks, four components of interaction will be confounded.
- The remaining unconfounded components of these interactions can be determined by subtracting the sum of squares for the confounded component from the sum of squares for the entire interaction

Block 1	Block 2	Block 3	Block 4	Block 5	Block 6	Block 7	Block 8	Block 9
0000	0001	2000	0200	0020	0010	1000	0100	0002
0122	0120	2122	0022	0112	0102	1122	0222	0121
0211	0212	2211	0111	0201	0221	1211	0011	0210
1021	1022	0021	1221	1011	1001	2021	1121	1020
1110	1111	0110	1010	1100	1120	2110	1210	1112
1202	1200	0202	1102	1222	1212	2202	1002	1201
2012	2010	1012	2212	2002	2022	0012	2112	2011
2101	2102	1101	2001	2121	2111	0101	2201	2100
2220	2221	1220	2120	2210	2200	0220	2020	2222
(L_1, L_2) = (0,0)	(0,2)	(2,2)	(2,1)	(2,0)	(1,6)	(1,1)	(1,2)	(0,1)

■ **FIGURE 9.8** **The 3^4 design in nine blocks with ABC, AB^2D^2, AC^2D, and BC^2D^2 confounded**

The 3^k Factorial Design in 3^p Blocks

- The 3^k factorial design may be confounded in 3^p blocks of 3^{k-p} observations each, where $p < k$.
- The procedure is to select p independent effects to be confounded with blocks. As a result, exactly $(3^p - 2p - 1)/2$ other effects are automatically confounded.
- These effects are the generalized interactions of those effects originally chosen.
- Let us consider an example where 3^7 design to be confounded in 27 blocks

- Because p = 3, we would select three independent components of interaction and automatically confound $[3^3 - 2(3) - 1]/2 = 10$ others.
- Suppose we choose ABC^2DG, BCE^2F^2G, and BDEFG. Three defining contrasts can be constructed from these effects, and the 27 blocks can be generated by the methods previously described.
- The other 10 effects confounded with blocks are mentioned in the next slide

$$(ABC^2DG)(BCE^2F^2G) = AB^2DE^2F^2G^2$$

$$(ABC^2DG)(BCE^2F^2G)^2 = AB^3C^4DE^4F^4G^3 = ACDEF$$

$$(ABC^2DG)(BDEFG) = AB^2C^2D^2EFG^2$$

$$(ABC^2DG)(BDEFG)^2 = AB^3C^2D^3E^2F^2G^3 = AC^2E^2F^2$$

$$(BCE^2F^2G)(BDEFG) = B^2CDE^3F^3G^2 = BC^2D^2G$$

$$(BCE^2F^2G)(BDEFG)^2 = B^3CD^2E^4F^4G^3 = CD^2EF$$

$$(ABC^2DG)(BCE^2F^2G)(BDEFG) = AB^3C^3D^2E^3F^3G^3 = AD^2$$

$$(ABC^2DG)^2(BCE^2F^3G)(BDEFG) = A^2B^4C^5D^3G^4 = AB^2CG^2$$

$$(ABC^2DG)(BCE^2F^2G)^2(BDEFG) = ABCD^2E^2F^2G$$

$$(ABC^2DG)(BCE^2F^2G)(BDEFG)^2 = ABC^3D^3E^4F^4G^4 = ABEFG$$

- This is a huge design requiring $3^7 = 2187$ observations arranged in 27 blocks of 81 observations each.

Fractional Replication of the 3^k Factorial Design-The One-Third Fraction of the 3k Factorial Design

The largest fraction of the 3^k design is a one-third fraction containing 3^{k-1} runs. Consequently, we refer to this as a 3^{k-1} fractional factorial design. To construct a 3^{k-1} fractional factorial design, select a two-degrees-of-freedom component of interaction (generally, the highest order interaction) and partition the *full* 3^k design into three blocks. Each of the three resulting blocks is a 3^{k-1} fractional design, and any one of the blocks may be selected for use. If $AB^{\alpha_2}C^{\alpha_3}\cdots K^{\alpha_k}$ is the component of interaction used to define the blocks, then $I = AB^{\alpha_2}C^{\alpha_3}\cdots K^{\alpha_k}$ is called the **defining relation** of the fractional factorial design. Each main effect or component of interaction estimated from the 3^{k-1} design has two aliases, which may be found by multiplying the effect by *both* I and I^2 modulus 3.

- Consider a one-third fraction of the 3^3 design.
- We may select any component of the ABC interaction to construct the design, that is, ABC, AB^2C, ABC^2, or AB^2C^2.
- Thus, there are actually 12 different one-third fractions of the 3^3 design defined by

$$x_1 + \alpha_2 x_2 + \alpha_3 x_3 = u \pmod 3 \quad \text{where } \alpha = 1 \text{ or } 2 \text{ and } u = 0, 1, \text{ or } 2$$

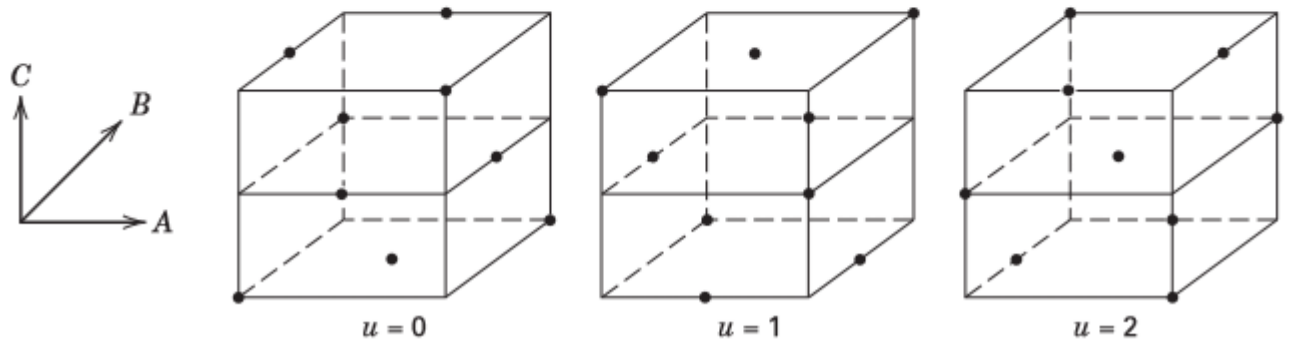
- Suppose we select the component of AB^2C^2. Each fraction of the resulting 3^{3-1} design will contain exactly $3^2 = 9$ treatment combinations that must satisfy

$$x_1 + 2x_2 + 2x_3 = u \pmod 3$$

  - where u =0,1 or 2

Design 1 $u = 0$	Design 1 $u = 1$	Design 1 $u = 2$
000	100	200
012	112	212
101	201	001
202	002	102
021	121	221
110	210	010
122	222	022
211	011	111
220	020	120

(*a*) Treatment combinations



(*b*) Geometric view

■ **FIGURE 9.9** The three one-third fractions of the 3^3 design with defining relation $I = AB^2C^2$

If any one of the 3^{3-1} designs in Figure 9.9 is run, the resulting alias structure is

$$A = A(AB^2C^2) = A^2B^2C^2 = ABC$$
$$A = A(AB^2C^2)^2 = A^3B^4C^4 = BC$$
$$B = B(AB^2C^2) = AB^3C^2 = AC^2$$
$$B = B(AB^2C^2)^2 = A^2B^5C^4 = ABC^2$$
$$C = C(AB^2C^2) = AB^2C^3 = AB^2$$
$$C = C(AB^2C^2)^2 = A^2B^4C^5 = AB^2C$$
$$AB = AB(AB^2C^2) = A^2B^3C^2 = AC$$
$$AB = AB(AB^2C^2)^2 = A^3B^5C^4 = BC^2$$

Consequently, the four effects that are actually estimated from the eight degrees of freedom in the design are $A + BC + ABC$, $B + AC^2 + ABC^2$, $C + AB^2 + AB^2C$, and $AB + AC + BC^2$. This design would be of practical value only if all the interactions were small relative to the main effects. Because the main effects are aliased with two-factor interactions, this is a resolution III design. Notice how complex the alias relationships are in this design. Each main effect is aliased with a *component* of interaction. If, for example, the two-factor interaction BC is large, this will potentially distort the estimate of the main effect of A and make the $AB + AC + BC^2$ effect very difficult to interpret. It is very difficult to see how this design could be useful unless we assume that all interactions are negligible.

- Let us consider the design with u=0 (Fig 9.9, present one slide before)
- If we let A denote the row and B denote the column, then the design can be written as

000	012	021
101	110	122
202	211	220

- The assumption of negligible interactions required for unique interpretations of the design is paralleled in the Latin square design.
- However, the two designs arise from different motives, one as a consequence of fractional replication and the other from randomization restrictions which is a 3 * 3 Latin square.
- The assumption of negligible interactions required for unique interpretations of the 3^{3-1}_{III} design is paralleled in the Latin square design. However, the two designs arise from different motives, one as a consequence of fractional replication and the other from randomization restrictions.

The treatment combinations in a 3^{k-1} design with the defining relation $I = AB^{\alpha_2}C^{\alpha_3}\cdots K^{\alpha_k}$ can be constructed using a method similar to that employed in the 2^{k-p} series. First, write down the 3^{k-1} runs for a **full** three-level factorial design in $k-1$ factors, with the usual 0, 1, 2 notation. This is the **basic design** in the terminology of Chapter 8. Then introduce the kth factor by equating its levels x_k to the appropriate component of the highest order interaction, say $AB^{\alpha_2}C^{\alpha_3}\cdots(K-1)^{\alpha_{k-1}}$, through the relationship

$$x_k = \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_{k-1} x_{k-1} \qquad \textbf{(9.5)}$$

where $\beta_i = (3-\alpha_k)\alpha_i \pmod 3$ for $1 \le i \le k-1$. This yields a design of the highest possible resolution.

As an illustration, we use this method to generate the 3_{IV}^{4-1} design with the defining relation $I = AB^2CD$ shown in Table 9.6. It is easy to verify that the first three digits of each treatment combination in this table are the 27 runs of a full 3^3 design. This is the basic design. For AB^2CD, we have $\alpha_1 = \alpha_3 = \alpha_4 = 1$ and $\alpha_2 = 2$. This implies that $\beta_1 = (3 - 1)\alpha_1 \pmod 3 = (3 - 1)(1) = 2$, $\beta_2 = (3 - 1)\alpha_2 \pmod 3 = (3 - 1)(2) = 4 = 1 \pmod 3$, and $\beta_3 = (3 - 1)\alpha_3 \pmod 3 = (3 - 1)(1) = 2$. Thus, Equation 9.5 becomes

$$x_4 = 2x_1 + x_2 + 2x_3 \tag{9.6}$$

The levels of the fourth factor satisfy Equation 9.6. For example, we have $2(0) + 1(0) + 2(0) = 0$, $2(0) + 1(1) + 2(0) = 1$, $2(1) + 1(1) + 2(0) = 3 = 0$, and so on.

■ **TABLE 9.6**
A 3_{IV}^{4-1} Design with $I = AB^2CD$

0000	0012	2221
0101	0110	0021
1100	0211	0122
1002	1011	0220
0202	1112	1020
1201	1210	1121
2001	2010	1222
2102	2111	2022
2200	2212	2120

The resulting 3_{IV}^{4-1} design has 26 degrees of freedom that may be used to compute the sums of squares for the 13 main effects and components of interactions (and their aliases). The aliases of any effect are found in the usual manner; for example, the aliases of A are $A(AB^2CD) = ABC^2D^2$ and $A(AB^2CD)^2 = BC^2D^2$. One may verify that the four main effects are clear of any two-factor interaction components, but that some two-factor interaction components are aliased with each other. Once again, we notice the complexity of the alias structure. If any two-factor interactions are large, it will likely be very difficult to isolate them with this design.

Other 3^{k-p} Fractional Factorial Designs

For moderate to large values of k, even further fractionation of the 3^k design is potentially desirable. In general, we may construct a $(\frac{1}{3})^p$ fraction of the 3^k design for $p < k$, where the fraction contains 3^{k-p} runs. Such a design is called a 3^{k-p} fractional factorial design. Thus, a 3^{k-2} design is a one-ninth fraction, a 3^{k-3} design is a one-twenty-seventh fraction, and so on.

The procedure for constructing a 3^{k-p} fractional factorial design is to select p components of interaction and use these effects to partition the 3^k treatment combinations into 3^p blocks. Each block is then a 3^{k-p} fractional factorial design. The defining relation I of any fraction consists of the p effects initially chosen and their $(3^p - 2p - 1)/2$ generalized interactions. The alias of any main effect or component of interaction is produced by multiplication modulus 3 of the effect by I and I^2.

We may also generate the runs defining a 3^{k-p} fractional factorial design by first writing down the treatment combinations of a full 3^{k-p} factorial design and then introducing the additional p factors by equating them to components of interaction

We illustrate the procedure by constructing a 3^{4-2} design, that is, a one-ninth fraction of the 3^4 design. Let AB^2C and BCD be the two components of interaction chosen to construct the design. Their generalized interactions are $(AB^2C)(BCD) = AC^2D$ and $(AB^2C)(BCD)^2 = ABD^2$. Thus, the defining relation for this design is $I = AB^2C = BCD = AC^2D = ABD^2$, and the design is of resolution III. The nine treatment combinations in the design are found by writing down a 3^2 design in the factors A and B, and then adding two new factors by setting

$$x_3 = 2x_1 + x_2$$

$$x_4 = 2x_2 + 2x_3$$

This is equivalent to using AB^2C and BCD to partition the full 3^4 design into nine blocks and then selecting one of these blocks as the desired fraction. The complete design is shown in Table 9.7.

■ **TABLE 9.7**

A 3^{4-2}_{III} Design with $I = AB^2C$ and $I = BCD$

0000	0111	0222
1021	1102	1210
2012	2120	2201

This design has eight degrees of freedom that may be used to estimate four main effects and their aliases. The aliases of any effect may be found by multiplying the effect modulus 3 by AB^2C, BCD, AC^2D, ABD^2, and their squares. The complete alias structure for the design is given in Table 9.8.

From the alias structure, we see that this design is useful only in the absence of interaction. Furthermore, if A denotes the rows and B denotes the columns, then from examining Table 9.7 we see that the 3^{4-2}_{III} design is also a Graeco–Latin square.

■ **TABLE 9.8**

Alias Structure for the 3^{4-2}_{III} Design in Table 9.7

Effect	Aliases							
	I				I^2			
A	ABC^2	$ABCD$	ACD^2	AB^2D	BC^2	$AB^2C^2D^2$	CD^2	BD^2
B	AC	BC^2D^2	ABC^2D	AB^2D^2	ABC	CD	AB^2C^2D	AD^2
C	AB^2C^2	BC^2D	AD	$ABCD^2$	AB^2	BD	ACD	ABC^2D^2
D	AB^2CD	BCD^2	AC^2D^2	AB	AB^2CD^2	BC	AC^2	ABD

Factorials with Mixed Levels- Factors at Two and Three Levels

- Consider a design that has some factors at two levels and some factors at three levels
- Due to some constraints, we are considering fractional factorial design
- If all of the factors are quantitative, mixed-level fractions are usually poor alternatives to a 2^{k-p} fractional factorial with center points.
- Usually when these designs are considered, the experimenter has a mix of qualitative and quantitative factors, with the qualitative factors taking on three levels.
- The complex aliasing we observed in the 3^{k-p} design with qualitative factors carries over to a great extent in the mixed-level fractional system.
- Thus, mixed-level fractional designs with all qualitative factors or a mix of qualitative and quantitative factors should be used very carefully.

- Suppose we have two variables, with A at two levels and X at three levels.
- Consider a table of plus and minus signs for the usual eight-run 2^3 design. The signs in columns B and C have the pattern shown on the left side of Table 9.9.

■ TABLE 9.9

Use of Two-Level Factors to Form a Three-Level Factor

Two-Level Factors		Three-Level Factor
B	*C*	*X*
−	−	x_1
+	−	x_2
−	+	x_2
+	+	x_3

- Let the levels of X be represented by x1, x2, and x3.
- The right side of Table 9.9 shows how the sign patterns for B and C are combined to form the levels of the three-level factor.
- Now factor X has two degrees of freedom and if the factor is quantitative, it can be partitioned into a linear and a quadratic component, each component having one degree of freedom.

- Table 9.10(next slide) shows a 2^3 design with the columns labeled to show the actual effects that they estimate, with X_L and X_Q denoting the linear and quadratic effects of X, respectively.
- Note that the linear effect of X is the sum of the two effect estimates computed from the columns usually associated with B and C and that the effect of A can only be computed from the runs where X is at either the low or high levels, namely, runs 1, 2, 7, and 8.
- Similarly, the A * X_L effect is the sum of the two effects that would be computed from the columns usually labeled AB and AC. Furthermore, note that runs 3 and 5 are replicates. Therefore, a one-degree-of-freedom estimate of error can be made using these two runs.
- Similarly, runs 4 and 6 are replicates, and this would lead to a second one-degree-of-freedom estimate of error.

One 2-Level and One 3-Level Factor in a 2^3 Design

	A	X_L	X_L	$A \times X_L$	$A \times X_L$	X_Q	$A \times X_Q$	Actual Treatment Combinations	
Run	A	B	C	AB	AC	BC	ABC	A	X
1	−	−	−	+	+	+	−	Low	Low
2	+	−	−	−	−	+	+	High	Low
3	−	+	−	−	+	−	+	Low	Med
4	+	+	−	+	−	−	−	High	Med
5	−	−	+	+	−	−	+	Low	Med
6	+	−	+	−	+	−	−	High	Med
7	−	+	+	−	−	+	−	Low	High
8	+	+	+	+	+	+	+	High	High

- The average variance at these two pairs of runs could be used as a mean square for error with two degrees of freedom.

■ **TABLE 9.11**

Analysis of Variance for the Design in Table 9.10

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square
A	SS_A	1	MS_A
X $(X_L + X_Q)$	SS_X	2	MS_{X_1}
AX $(A \times X_L + A \times X_Q)$	SS_{AX}	2	MS_{AX}
Error (from runs 3 and 5 and runs 4 and 6)	SS_E	2	MS_E
Total	SS_T	7	

If we are willing to assume that the two-factor and higher interactions are negligible, we can convert the design in Table 9.10 into a resolution III fraction with up to four two-level factors and a single three-level factor. This would be accomplished by associating the two-level factors with columns *A, AB, AC*, and *ABC*. Column *BC* cannot be used for a two-level factor because it contains the quadratic effect of the three-level factor *X*.

This same procedure can be applied to the 16-, 32-, and 64-run 2^k designs. For 16 runs, it is possible to construct resolution V fractional factorials with two two-level factors and either two or three factors at three levels. A 16-run resolution V fraction can also be obtained with three two-level factors and one three-level factor. If we include 4 two-level factors and a single three-level factor in 16 runs, the design will be of resolution III. The 32- and 64-run designs allow similar arrangements.

Factors at Two and Four Levels

It is very easy to accommodate a four-level factor in a 2^k design. The procedure for doing this involves using two two-level factors to represent the four-level factor. For example, suppose that A is a four-level factor with levels a_1, a_2, a_3, and a_4. Consider two columns of the usual table of plus and minus signs, say columns P and Q. The pattern of signs in these two columns is as shown on the left side of Table 9.12. The right side of this table shows how these four sign patterns would correspond to the four levels of factor A. The effects represented by columns P and Q and the PQ interaction are mutually orthogonal and correspond to the three-degrees-of-freedom A effect. This method of constructing a four-level factor from two two-level factors is called the method of **replacement**.

To illustrate this idea more completely, suppose that we have one four-level factor and two two-level factors and that we need to estimate all the main effects and interactions involving these factors. This can be done with a 16-run design. Table 9.13 shows the usual table of plus and minus signs for the 16-run 2^4 design, with columns A and B used to form the four-level factor, say X, with levels x_1, x_2, x_3, and x_4. Sums of squares would be calculated for each column $A, B, \ldots, ABCD$ just as in the usual 2^k system. Then the sums of squares for all factors X, C, D, and their interactions are formed as follows:

$$SS_X = SS_A + SS_B + SS_{AB} \quad \text{(3 degrees of freedom)}$$

$$SS_C = SS_C \quad \text{(1 degree of freedom)}$$

$$SS_D = SS_D \quad \text{(1 degree of freedom)}$$

$$SS_{CD} = SS_{CD} \quad \text{(1 degree of freedom)}$$

$$SS_{XC} = SS_{AC} + SS_{BC} + SS_{ABC} \quad \text{(3 degrees of freedom)}$$

$$SS_{XD} = SS_{AD} + SS_{BD} + SS_{ABD} \quad \text{(3 degrees of freedom)}$$

$$SS_{XCD} = SS_{ACD} + SS_{BCD} + SS_{ABCD} \quad \text{(3 degrees of freedom)}$$

■ TABLE 9.13

A Single Four-Level Factor and Two Two-Level Factors in 16 Runs

Run	(A	B)	= X	C	D	AB	AC	BC	ABC	AD	BD	ABD	CD	ACD	BCD	ABCD
1	−	−	x_1	−	−	+	+	+	−	+	+	−	+	−	−	+
2	+	−	x_2	−	−	−	−	+	+	−	+	+	+	+	−	−
3	−	+	x_3	−	−	−	+	−	+	+	−	+	+	−	+	−
4	+	+	x_4	−	−	+	−	−	−	−	−	−	+	+	+	+
5	−	−	x_1	+	−	+	−	−	+	+	+	−	−	+	+	−
6	+	−	x_2	+	−	−	+	−	−	−	+	+	−	−	+	+
7	−	+	x_3	+	−	−	−	+	−	+	−	+	−	+	−	+
8	+	+	x_4	+	−	+	+	+	+	−	−	−	−	−	−	−
9	−	−	x_1	−	+	+	+	+	−	−	−	+	−	+	+	−
10	+	−	x_2	−	+	−	−	+	+	+	−	−	−	−	+	+
11	−	+	x_3	−	+	−	+	−	+	−	+	−	−	+	−	+
12	+	+	x_4	−	+	+	−	−	−	+	+	+	−	−	−	−
13	−	−	x_1	+	+	+	−	−	+	−	−	+	+	−	−	+
14	+	−	x_2	+	+	−	+	−	−	+	−	−	+	+	−	−
15	−	+	x_3	+	+	−	−	+	−	−	+	−	+	−	+	−
16	+	+	x_4	+	+	+	+	+	+	+	+	+	+	+	+	+

This could be called a 4×2^2 design. If we are willing to ignore two-factor interactions, up to nine additional two-level factors can be associated with the two-factor interaction (except AB), three-factor interaction, and four-factor interaction columns.

There are a wide range of fractional factorial designs with a mix of two- and four-level factors available. However, we recommend using these designs cautiously. If all factors are quantitative, the 2^{k-p} system with center points will usually be a superior alternative. Designs with factors at two and four levels that are of resolution IV or higher, which would usually be necessary if there are both quantitative and qualitative factors present, and typically rather large, requiring $n \geq 32$ runs in many cases.